Days and Dates

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A few years ago, I was trying to figure out what days of the week my birthdays would fall on during college years. I noticed that I had a birthday on a Thursday, and the next year on a Friday. The next year, however, Saturday was skipped, and my birthday was a Sunday. In fact, the next time my birthday was on a Saturday wasn't for more almost 10 years. This made me wonder how many years the "cycle of days that a date fall on" is.

When I say I want to know the "cycle of days that a date falls on", I mean I am looking for the number of years it takes before the day of the week a day falls on repeats. I will explain this further with an example. Imagine another world, with three weekdays, oneday, twoday and threeday. We can label the days with numbers, 1, 2 and 3. In this world, a year is defined so that the day a date falls on moves from 1 to 2, 2 to 3, and 3 to 1. In other words, we can describe the "cycle of the days of the week that a date falls on" by

$$(1, 2, 3).$$

Since this cycle is 3 elements long, it takes three years for the days a date falls on to repeat.

Now we shall look at the real world. Looking at a regular ole year, it becomes apparent that each year the day of the week a date falls on moves forward by 1. This gives us that it takes 7 years for the days a date falls on to repeat. But wait! What about leap years? That's right, leap years throw everything off. As most people know, leap years occur every 4 years. However, what many people don't realize is there are more rules about leap years. On years that are divisible by 100, even though the year is divisible by 4, there is no leap year. However, if the year is divisible by 400, it is a leap year after all. These extra rules are to keep the summer solstice happening around June 21 each year, and wikipedia has a leap year page with a great explanation of this.

Now that we are aware of leap years, we can temporarily forget about the extra rules, and just pretend they are every 4 years. Let us find how long it takes for the day of a date to cycle under this assumption. We label the days of the week 0-6. (Note that it does not matter which day is considered 0, as long as the days are in order). Now we must look at what day a date falls on each year. Let's imagine we start at year 0, choosing a date on a day 0. Then in year 1 the day will be 2, since year 0 was a leap year, and in year 2 the day will be 3. We can make a table to see how the date's day progresses:

Year:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Day:	0	2	3	4	5	0	1	2	3	5	6	0	1	3	4

Year:	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Day:	5	6	1	2	3	4	6	0	1	2	4	5	6	0

As we can see, in year 5 the day of the week for our chosen date is once again 0, however this is not the end of the cycle, as the next few terms of the sequence do not match the beginning few terms. This is because year 5 is a different distance from the next leap year from year 0. In other words,

$$0 \not\equiv 5 \mod 4$$

If we find a day 0 in a year that is congruent to $0 \mod 4$ then we will be guaranteed to have the cycle repeat, since the days will undergo the same shifts due to leap years as we progress farther. Examining the table, we find that this happens in year 28. Thus we see that the day of the week a date falls on under the 4-year leap year rule has a 28 year cycle. Or at least we think it does. For this table, we started in a year that was congruent to $0 \mod 4$. What happens if we start in years congruent to 1, 2 or $3 \mod 4$? I have included tables for these in the appendix, but it does turn out that the cycle repeats after 28 years, regardless of the start year.

This is great and all, but how does it help us? Let's look at the 400 year leap year cycle, which accounts for all rules concerning leap years. We once again start from year 0, on day 0. Noting that 392 is a multiple of 28 ($392 = 28 \cdot 14$), we see the day in year 392 would be 0 by the 4-year leap year rule? However, in those 392 years, three leap years were missed at years 100, 200 and 300, so we are actually 3 days behind, putting the day of the date in year 392 at day 4. When we reach year 400, 8 years will have passed. Looking at any of our tables displaying the 28 year 4-year leap year cycle, we see 8 years will add 3 days. Thus, the date in year 400 will be day 0.

To generalize, we can pick any date, regardless of the year or day of the week. By the 4-year leap year rule, though the cycle may change depending on the year, after every 28 years the day of the week for the date will cycle. Regardless of the start year, the 4-year leap year rule will cause the day of the date in 400 years to be 3 days ahead of the start day (the same after 392, then 3 ahead after 8 more). However, there are 3 years divisible by 100 during these 400 years. On these years, a leap year by the 4-leap year rule doesn't actually happen, which means we have counted 3 non-existent leap days. Therefore our day 3 result in the 400th year is 3 days ahead, thus the true day of the date is day 0. Since leap years are defined exactly the same in the next 400 years, this cycle is guaranteed to repeat. (Note that the cycle cannot be less than 400 years, since the cycle of leap years is 400 years. Any smaller cycle will contain leap years in different spots after some number of cycles).

Appendix

Day of the week cycle starting in a year 1 mod 4:

Year:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Day:	0	1	2	3	5	6	0	1	3	4	5	6	1	2	3

Year:	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Day:	4	6	0	1	2	4	5	6	0	2	3	4	5	0

Day of the week cycle starting in a year 2 mod 4:

Year:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Day:	0	1	2	4	5	6	0	2	3	4	5	0	1	2	3

Year:	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Day:	5	6	0	1	3	4	5	6	1	2	3	4	6	0

Day of the week cycle starting in a year 3 mod 4:

Year:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Day:	0	1	3	4	5	6	1	2	3	4	6	0	1	2	4

Year:	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Day:	5	6	0	2	3	4	5	0	1	2	3	5	6	0