A Mathematical View of the Rubik's Cube and Other Twisty Puzzles

Kenneth R Roffo Jr
B.S. in Physics, Computer Science, Mathematics

State University of New York, College at Oswego
College Honors Program

September, 2016

Abstract

Rubik's Cube is a very well-known puzzle which looks simple at first, but turns out to be very difficult. There are many other puzzles similar to Rubik's Cube called twisty puzzles. Twisty puzzles come in all sorts of different shapes and sizes, but they all exhibit very interesting mathematical properties. In this project I have analyzed the Rubik's Cube as well as other twisty puzzles from a mathematical standpoint. The Rubik's Cube is shown to be solvable using only five of its six side turns. Mathematical constructs, called groups, are presented and used to understand how twisty puzzles behave. The Dino Cube, a corner turning twisty puzzle, is lightly explored. Finally, an original cube created during the project, now called the Roffo Cube, is examined.

Contents

1 Advice to Future Honors Thesis Students

The Honors Thesis scares a lot of students into dropping out of the Honors Program at SUNY Oswego; I have witnessed at least a dozen people drop the program because of the thesis, well before even starting the project. While it is true that the thesis is a lot of work, it is false that the thesis is something that you will dread doing while you work on it. It is not required so that you do more work, but because it truly is a great experience. The honors thesis allows you to explore a topic of your interest, and truly do what you want. It is not an assignment in which you must do something you don't care about; it is an opportunity for you to do something amazing. That being said, here is some advice for working on your thesis.

Pick a topic you are very interested in. You will be working on the project for a long time, and you will have a much better experience doing something that interests you rather than something you think looks good to other people. This is the most important cubie of advice, and you will hear it many times.

Start early. The earlier you start, the more time you have. With more time you will be less stressed about meeting deadlines, and you will be more focused on learning rather than actually putting together a paper about the project.

Set goals. Having a long time to do a project, it can be easy to get into the mindset that there is no rush to get things done. This can be very bad, as it can lead you to make little to no progress in the beginning or middle of your project. You should meet with your adviser and set goals for when you would like to have certain things done. It can be very helpful to have regularly scheduled meetings, with goals from one to the next. I personally did weekly meetings, and had a new set of goals every week.

2 Acknowledgments

First of all I would like to thank my primary adviser Dr. Graham for all the time and effort she has put into this project. We worked on it for almost two years, and I learned so much about how math research works. Dr. Graham has also helped me to develop my presentation skills through several presentations I have given both at two Quests (SUNY Oswego's annual research presentation day), and Math Conferences. She has been an outstanding adviser, and I thank her dearly.

I would also like to thank Dr. Vampola who has been a very supportive secondary adviser, and overall a great mentor. Dr. Vampola has opened my eyes to many things I had not before considered about the world, and the things within it.

The SCAC grant has been an amazing help in paying for all the 3D printing involved in the project, and its providers deserve a huge thank you for that! Thank you!

Sharona Ginsberg at the 3D printing office in Penfield has been an amazing help. She did all the printing for the project, and she has also been very supportive of the project. Thank you Sharona!

Lastly, I would like to thank the Honors Program at SUNY Oswego for providing me the opportunity to work on such a great project. It was an extremely meaningful experience, and I truly believe it is an honor to have been a member of the program.

3 Author's Reflections

As a freshman, hearing about the Honors Thesis was scary. Many people dropped out of the honors program because they didn't want that kind of responsibility, and many of the students who stayed in the honors program dreaded having to work on the thesis eventually. I never complained about having to do the project, but I will admit that there were times where I worried about how much work it would take. The Honors Thesis did take a lot of work. A lot. However, having worked on it for almost two years, I can honestly say that it was not something painful or boring. The process of studying something I love through a subject I love was pretty enlightening.

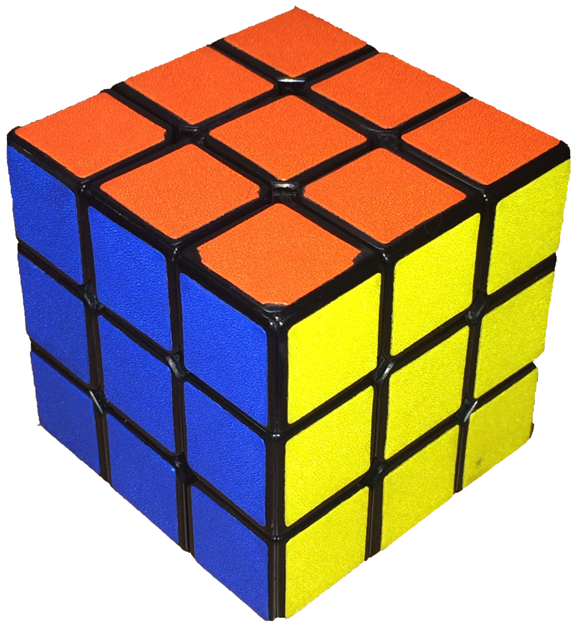
Having weekly meetings for three and a half semesters on this project, I have certainly learned a lot about math and twisty puzzles which I never even talked about in the classroom. This alone makes the project an awesome opportunity, but the benefits don't stop there. Presenting on the project at conferences made me a better presenter. Explaining the project to other students and professors made me better at putting my thoughts and understanding of complex things into words that inexperienced minds could understand.

Now that I am done with the project, I feel that I have grown both as a student, and as a person. The Thesis was never something to dread; it was something to look forward to, an opportunity.

4 Thesis Body

Introduction

One of the most fascinating puzzles available today is the Rubik's Cube. The Rubik's Cube is a cube whose 6 sides are made up of smaller pieces called cubies. Cubies can move around each other such that each side of the puzzle can be rotated about its center. Each of these sides is referred to as a face, and each face contains 9 square stickers. When the puzzle is solved, each face contains stickers of only one color.



In total, there are 54 stickers that must be put in the correct spots for the puzzle to be solved. It has been calculated that there are over 43 quintillion different possible arrangements of the cubies. (Numberphile 2012)

This sounds daunting, but there is a much better way of looking at the cube. To start, we shall label the faces. Holding the cube looking directly at one face, the faces are labeled as follows:

The F face: The face in view, the *front* face.

The R face: The face to the right, the *right* face.

The L face: The face to the left, the *left* face.

The U face: The face on top, the *up* face.

The D face: The face on bottom, the *down* face.

The B face: The unseen face, the *back* face.

The one letter abbreviations will be our main way of referring to each face. Each of these faces consists of three types of cubies: centers, edges, and corners.

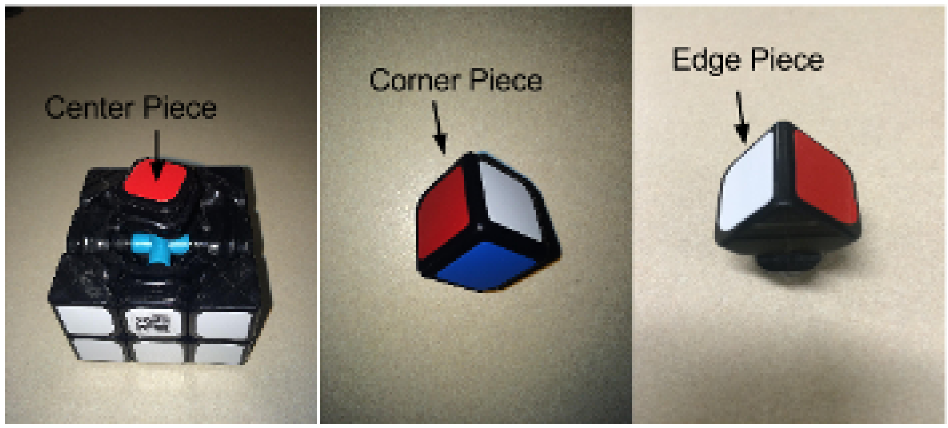


Figure 1: The U center, UFL Corner, and UF Edge cubies of a Rubik's Cube oriented with white in front and red on top.

1. There are 6 center cubies on the cube, each located in the center of a face. These cubies rotate in place, but they have fixed position relative to the other centers. This is helpful as it means the center cubies act as a reference for where other cubies belong. Center cubies have one sticker. They determine the color that the corresponding face will be when the cube is solved. A center can be referred to by the letter representing the face it is a part of. For example the center on the U face is the U center.

2. There are 12 edge cubies on the cube. The edge cubies are in the middle of the rows and columns along the sides of the cube. Edges have two stickers. An edge cubie can be referred to by the letters representing the faces it is a part of. For example, the UF edge is the edge that has stickers on both the U and F faces of the cube.

3. Lastly, there are 8 corner cubies on the cube. Corner cubies have 3 stickers. Like edges, corners are individually labeled by the three faces they have stickers on. The UFR corner is a member of the U, F, and R faces. When referring to a cubie, the order in which these letters appear does not matter. However, we will refer to specific corner stickers later and when doing so, the face the sticker is on should come first. Thus UFR, FUR and RUF all refer to the same corner, but they refer to three different stickers.

Now that we have an understanding of the different cubies, we can think of the cube as 20 movable cubies that must be put into a specific arrangement.

The cube supports one type of movement, called a face turn. A face turn is the 90 degree clockwise rotation of one face of the cube, so the edges and corners on that face move around the center. That means a face turn only affects cubies that are part of that face. Since there are six faces, there are only six possible clockwise 90 degree turns on the cube at any given time.

To keep track of this, we give names to each turn, the letter representing that face. A letter by itself means to turn the corresponding face 90 degrees clockwise. A B turn is turning the B face counterclockwise as viewed from the front of the cube, but it is actually turning clockwise if viewed from the back. A letter followed by a superscript 2, say B^2, means to turn the corresponding face 180 degrees. Lastly, a letter followed by a superscript -1, such as F^{-1}, means to turn the corresponding face counterclockwise. Thus to refer to the a face, such as the Right face, we say “the R face”, but to refer to an R turn, we say “R”.

Now that the notation for turns on the Rubik’s Cube is defined, we must define a scramble. A *scramble* is a sequence of turns. For example the scramble

$$FBF^2D^{-1}UR^2$$

means to execute an F turn, followed by a B turn, then an F^2 turn, then D^{-1}, U and finally R^2.

To find the *inverse* of a scramble, reverse its order and switch clockwise and counterclockwise rotations. The scramble

$$FBF^2D^{-1}UR^2$$

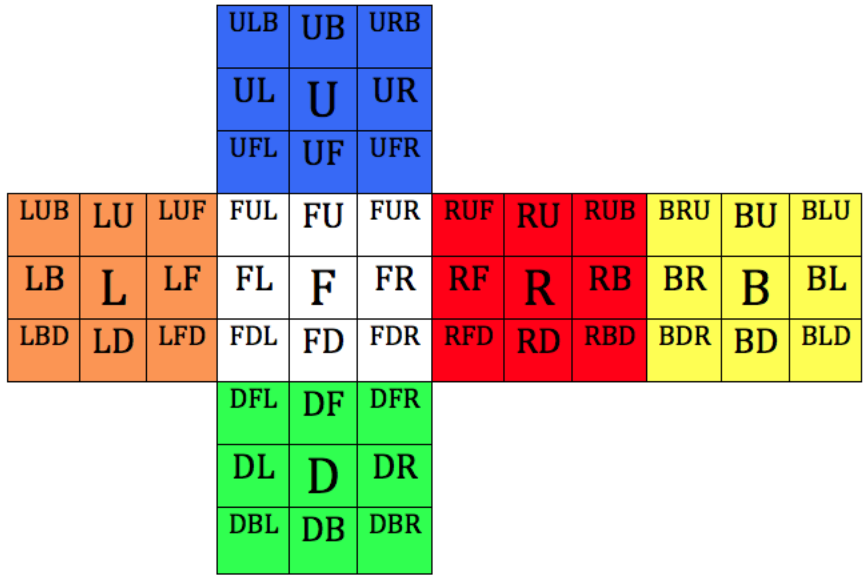
has inverse

$$R^2U^{-1}DF^2B^{-1}F^{-1}.$$

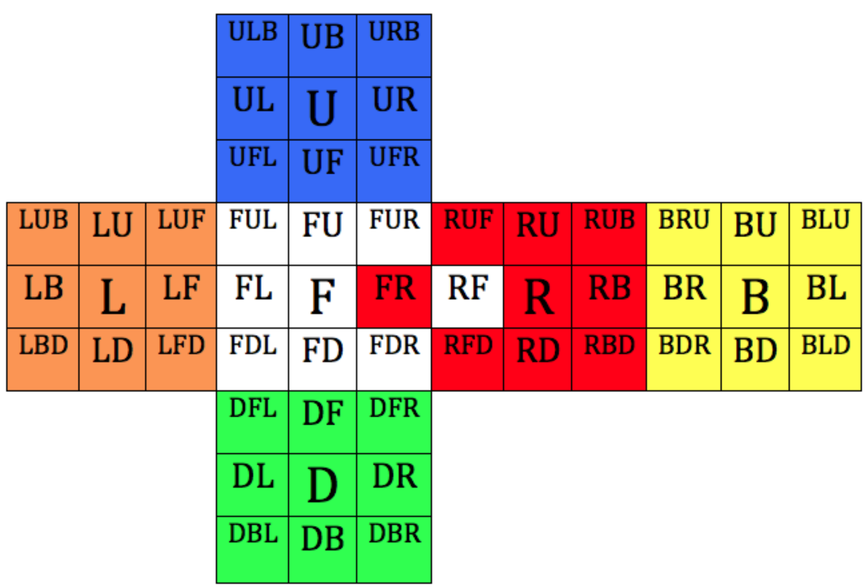
A solution to a scramble is a scramble that will return each cubie to the position in which it belongs. The inverse of a given scramble is also a solution to that scramble.

Permutations and Groups

Mathematicians have a practical way of describing things such as the behavior of twisty puzzles. These are called permutations. Basically, a face turn permutes, or changes the position of, each cubie on the turned face of the cube and thereby the stickers contained by those cubies. We can label the stickers, and write down how each face turn permutes the stickers of the cube. It is important to note that in the following diagram the labels refer to the positions the stickers are in, and not the stickers themselves.



Permutations can be written down using cycles such as (FR, RF). It is not standard to use commas in cycle notation, but we use them to help with clarity. In a cycle, the sticker in the position of a given symbol is sent to the position corresponding to the next symbol in the cycle (read left to right). Note that the last symbol's "next" is the first symbol in the cycle. In the cycle (FR, RF), the FR and RF stickers are swapped. Since these stickers are the stickers of the RF edge, this permutation corresponds to flipping the edge in it's place. The result of this permutation is shown below:



Cycle notation is very useful for studying twisty puzzles, as every turn on a twisty puzzle contains at least one cycle.

A permutation can have multiple cycles. Following are the 6 face turns written in cycle notation using the above labeling scheme:

$$
\begin{aligned}
U &= (UF,\ UL,\ UB,\ UR)(UFL,\ ULB,\ URB,\ UFR)(FU,\ LU,\ BU,\ RU)\\
&\quad (BRU,\ RUF,\ FUL,\ LUB)(BLU,\ RUB,\ FUR,\ LUF)\\
F &= (FD,\ FL,\ FU,\ FR)(FDL,\ FUL,\ FUR,\ FDR)(DF,\ LF,\ UF,\ RF)\\
&\quad (DFL,\ LUF,\ UFR,\ RFD)(DFR,\ LFD,\ UFL,\ RUF)\\
R &= (RD,\ RF,\ RU,\ RB)(RFD,\ RUF,\ RUB,\ RBD)(DR,\ FR,\ UR,\ BR)\\
&\quad (DFR,\ FUR,\ URB,\ BRD)(DBR,\ FDR,\ UFR,\ BRU)\\
L &= (LD,\ LB,\ LU,\ LF)(LBD,\ LUB,\ LUF,\ LFD)(FL,\ DL,\ BL,\ UL)\\
&\quad (BLD,\ ULB,\ FUL,\ DFL)(BLU,\ UFL,\ FDL,\ DBL)\\
D &= (DB,\ DL,\ DF,\ DR)(DBL,\ DFL,\ DFR,\ DBR)(BD,\ LD,\ FD,\ RD)\\
&\quad (BLD,\ RBD,\ FDR,\ LFD)(BDR,\ RFD,\ FDL,\ LBD)\\
B &= (BD,\ BR,\ BU,\ BL)(BDR,\ BRU,\ BLU,\ BLD)(DB,\ RB,\ UB,\ LB)\\
&\quad (RUB,\ DBR,\ LBD,\ ULB)(RBD,\ DBL,\ LUF,\ URB)
\end{aligned}
$$

Mathematicians describe permutations using the language of group theory.

Definition (Group)**.** A *group* $(G, *)$, where G is a set, and $*$ is a binary operation which combines two elements of G, and the following properties are satisfied:

1. $*$ is associative on G.
2. G is closed under $*$.
3. There exists an identity element e in G such that $g * e = e * g = g$, for all g in G.
4. For every g in G, there exists an inverse g^{-1} in G such that $g * g^{-1} = g^{-1} * g = e$.

Note that when multiple groups are present, say G and H, their operations may be referred to by $*_G$ for G and $*_H$ for H. (Dummit and Foote 2004)

For twisty puzzles, the set is the set of all scrambles. The operation is composition, or doing the permutation which appears on the left, then the one on the right. For simplicity, we will not write the $*$, thus $F * B = FB$, meaning an F turn, followed by a B turn (Note that this is in contrast to the standard form for composing permutations). For the remainder of the text, it should be clear based on context whether FB is referring to a permutation or a sticker.

It is important to understand that there are many ways to permute the stickers on a twisty puzzle, but there are (except for extremely simple twisty puzzles) fewer permutations in the group of the puzzle. There are permutations which are not possible to achieve using face turns. For example the permutation (F, UF) is impossible since it requires a center sticker to switch with an edge sticker. A less obvious example is that it is impossible to swap two edges on the cube (without also permuting other cubies). This means that the Rubik's Cube group is smaller than the group of permutations on the stickers. In fact, the Rubik's Cube group is a subgroup of the group of permutations on the stickers.

Definition (Subgroup)**.** A *subgroup* $(H, *)$ of a group $(G, *)$ is a group in which H is a subset of G (note that the operation must be the same in both the subgroup, H and the supergroup, G).

For example, the even integers under addition is a subgroup of all the integers under addition.

Definition (Generating Set)**.** A *generating set*, S for a group G is a set of elements from G such that for any g in G, g is the composition of a finite number of elements of S (or their inverses).

For example, the integers under addition has the generating set $\{1\}$. Every positive integer a can be created by adding 1 a times. Every negative integer b can be created by adding -1, the inverse of 1, b times. Lastly, 0 can be created by adding 1 to itself 0 times.

Note that generating sets are not unique, and in fact an entire group can be correctly referred to as a generating set for itself.

One way mathematicians like to study groups is by thinking of them as combinations of smaller, simpler groups. The direct product allows us to combine groups.

Definition (Direct Product)**.** A *direct product* of two groups G and H, denoted $G \times H$ is a group whose elements are all the ordered pairs (g, h) where $g \in G, h \in H$ and the operation is defined by

$$(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2).$$

Direct products can be formed with more than two groups. Doing so, this definition is extended by using tuples, as opposed to ordered pairs, where each element comes from the corresponding group in the product. (Burn 1985)

As an example, consider the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ ($\mathbb{Z}_n$ is shorthand for the group of integers under addition (mod n)). The direct product of the groups is $\mathbb{Z}_2 \times \mathbb{Z}_3$ and contains six elements:

$$(0,0),(0,1),(0,2),$$
$$(1,0),(1,1),(1,2).$$

The operation in this group will be a combination of addition mod 2 and mod 3. Combining some elements we see

$$\begin{aligned}(1,0)*(1,2) &= (1+1 \mod 2, 0+2 \mod 3)\\ &= (0,2).\end{aligned}$$

The Necessary and Sufficient Five Generators

Now that we have an understanding of some basic concepts in group theory, we can study the Rubik's Cube Group. The 6 turn types will now be referred to as the 6 generators, as they generate the Rubik's Cube group. This is assured by the fact that the only way to permute the cubies of the cube is by doing one of these 6 face turns. One might ask whether all 6 of the face turns are necessary to form a generating set. I claim that using no less than 5 generators, say F, U, R, L and B (no D turn), any scramble can be solved.

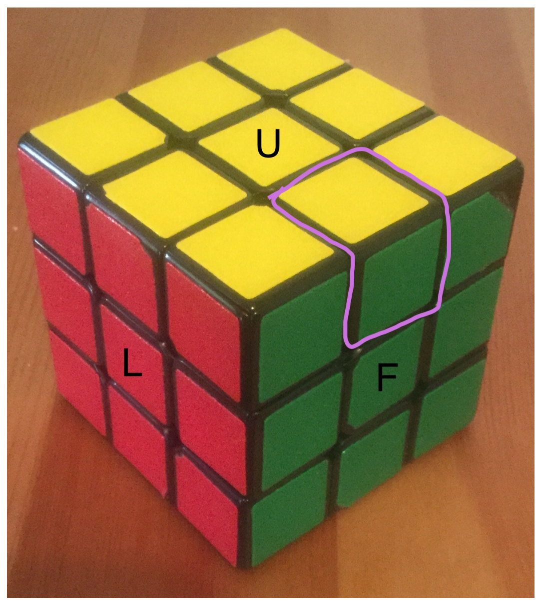
Proposition 1. Any 5 element subset of the set of face turns is a minimal generating set for the Rubik's Cube Group.

Proof. We must first show that it is possible to solve any scramble using 6 turn types. Every scramble is made using at most 6 turn types. The inverse of a scramble can be applied to return the cube to its original state (solved) and thus is a solution. Thus it is possible to solve any scramble using 6 turn types, F, L, R, U, B and D.

If it is possible to find a sequence of moves using 5 turn types that equals the sixth, then 5 turn types are sufficient to solve any scramble. Thus if we can construct a D move using the other 5 turn types, then any scramble is solvable using 5 turn types. Note that D can be executed twice for a D^2 and thrice for a D^{-1}, so only D must be constructed. The sequence of moves to simulate a D turn with the other 5 turn types is as follows (parentheses to help with clarity):

$$(R^2\ L^2\ U^{-1}\ F^2\ B^2)\ U^{-1}\ R^{-1}\ L\ U^2\ R^2\ L^2\ U^2\ R^2\ U^2\ R^2\ L^2\ U^2\ L$$
$$(R^{-1}\ U^2\ R^2\ U^2\ R^2\ U^2)\ (L^2\ U^2\ L^2\ U^2\ L^2\ U^2)\ (F^2\ U^2\ F^2\ U^2\ F^2\ U^2)$$
$$(B^2\ U^2\ B^2\ U^2\ B^2\ U^2)$$
$$(R\ U\ R^{-1}\ U^{-1}\ R^{-1}\ F\ R^2\ U^{-1}\ R^{-1}\ U^{-1}\ R\ U\ R^{-1}\ F^{-1})\ U$$
$$(R^{-1}\ F\ R^{-1}\ B^2\ R\ F^{-1}\ R^{-1}\ B^2\ R^2)\ U.$$

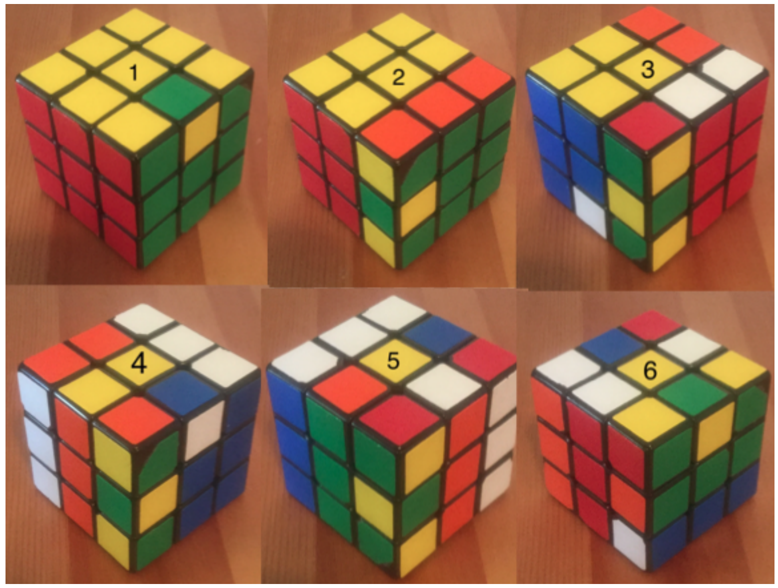
For any given scramble, a solution can be found using 6 turn types. Then, wherever a D appears, it can be replaced with the above permutation to give a solution which uses only 5 turn types.



We must now show that 5 turn types are necessary to solve any scramble. We will show it is not possible to solve every scramble using 4 turn types. To allow only 4 turn types, we must choose two turn types to not allow. Because of the symmetry of a cube, we can either choose two adjacent faces (Case 1), or two opposite faces (Case 2), to fix.

Case 1: Let's choose two adjacent faces to fix. Now orient the cube so that one of these faces is the Front face, and the other is the Up face. In our chosen orientation, the UF edge is only on the locked faces (Shown to the right). This means this cubie may never move using our available four turn types. Now consider the simple scramble U (remember our scramble is not limited to 4 turn types, only our solution). The obvious solution would be a U^{-1} turn, but we are not allowed to make F or U turns. We need a way to move the UF

edge back to its original, UR position. As stated above, since the UF edge is not a member of a movable face, this task is impossible, thus the cube cannot be solved.



Case 2: Now choose two opposite faces to fix. Without loss of generality, say the U and D faces are locked. Now consider any scramble in which the UF edge is in the right spot, but needs to be flipped to have the correct orientation. As an example we will refer to the above diagram.

In picture 1 we have a cube with a flipped edge in the UF position. The scramble I have used here is

$$D^2\,(R^{-1}\,L\,U^2\,R\,L^{-1}\,B\,L\,R^{-1}\,U^{-1}\,R\,L^{-1})\,F^2\,(R^{-1}\,L\,U\,R\,L^{-1}\,B^{-1}\,R^{-1}\,L\,U^2\,R\,L^{-1})\,F^2\,D^2$$

which results in the UF and BD edges each being flipped.

We need to be able to flip this edge and put it back in this spot without turning the U or D faces. When the edge is a part of the U or D faces it will be impossible to move it other than by turning the other face it is on, in this case F. Therefore moving the edge to the D or U faces will be useless and thus we shall not do so. This leaves us with 2 moves, F and F^{-1}, to choose from. However, due to the symmetry of the cube, one case will result in a flipped but equivalent version of the other, so we will just choose F^{-1}, knowing that F would lead to a similar outcome.

Now the edge is a member of the F and L faces as seen in picture 2. Since we decided to do an F^{-1} turn before, turning the F face now would contradict our decision, so we must choose an L turn. Note that L and L^{-1} would move the edge to the D and U faces respectively, which would force us to make another L turn, so the only useful move to help us reach our goal is L^2, which brings the cube to the state shown in picture 3 (the cube has been rotated in the pictures to show where the edge of interest is, but we will consider the Green face to still be the F face).

Now the edge is in the LB position. Just like before, we came from an L move, so another one would be pointless, and B and B^{-1} result in useless positions in the U and D layers. So B^2 must be the only sensible move to make here, which brings us to picture 4. For the same reasons the only move that makes sense is R^2 (since the edge is in the BR position), bringing us to picture 5.

Now the edge is once again a member of the F face. All thats left to do is an F^{-1} to put the edge back in the original position only to see that it has not been flipped, as seen in picture 6. This means that it is impossible for us to flip the edge without using a U or a D move. Therefore 4 turn types is not sufficient to solve every scramble.

Since 4 turn types is not enough to solve all scrambles, and 5 turn types is, it is indeed so that 5 turn types is necessary and sufficient to solve any scramble on a Rubiks cube. □

It is important to note that although the five face turns form a minimal generating set, they do not form the minimum generating set. In fact it has been shown that there exists a two-element generating set. These elements are, however, much more complex than the face turns. (Singmaster 1981)

Dino Cube Transitivity

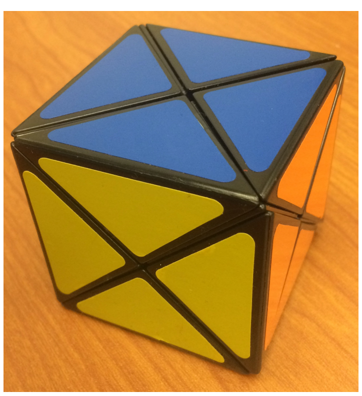


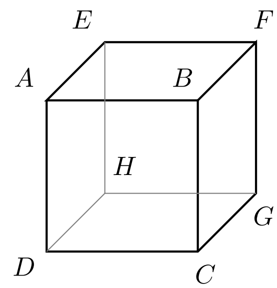
Figure 2: The Dino Cube, a corner turning twisty puzzle.

The Dino Cube is a twisty puzzle containing nothing more than twelve edges. The Dino Cube, compared to most twisty puzzles, is relatively simple. Sill we can study its behavior to learn something about more intense puzzles.

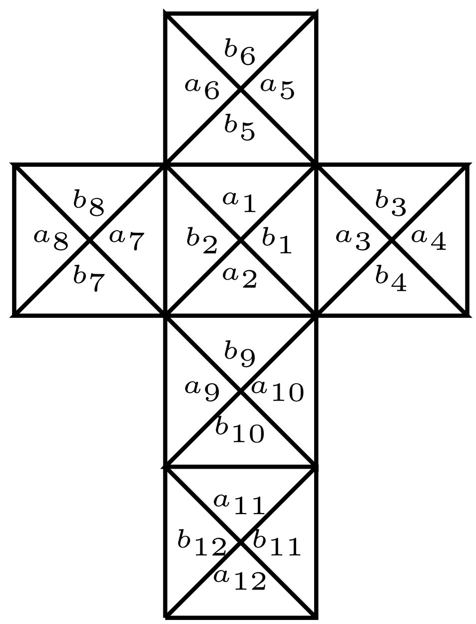
Unlike the Rubik's Cube, the Dino Cube does not twist around its six faces. Instead, the Dino Cube twists around its 8 corners. Since these 8 turns are the only way to move the cubies on the Dino Cube, they form a generating set for the Dino Cube group.

To aid us in our exploration of the Dino Cube, we will label the corners (and thereby, corner turns) using letters, just like the faces of the Rubik's Cube. The picture below displays the names of the corners we will use, A - H.

You may have noticed that each face of the Dino Cube is broken up into 4 stickers. We can also label the stickers to help us see how each twist of the puzzle acts on them. We use numbers to prevent confusion with the labeling of the corners.



The stickers have been labeled as shown in the diagram below. Here, the top square is the back face of the cube and the middle of the cross is the top face.



One of the things you may find yourself asking is if each sticker can go to the spot of every other sticker on the cube. If this is the case, we say the Dino Cube group is *transitive* on the stickers.

Proposition 2. The Dino Cube group is not transitive on the stickers.

Proof. We can begin to check if this is true by writing out the permutations. In what follows, the label of a corner represents the permutation on the stickers which occurs by a 120 degree clockwise rotation about that corner.

$$A = (a_2, a_9, a_7)(b_2, b_9, b_7)$$
$$B = (a_3, a_{10}, a_2)(b_2, b_4, b_9)$$
$$C = (a_{10}, a_4, a_{11})(b_4, b_{11}, b_{10})$$
$$D = (a_9, a_{11}, a_8)(b_7, b_{10}, b_{12})$$
$$E = (a_2, a_7, a_6)(b_2, b_8, b_5)$$
$$F = (a_1, a_5, a_3)(b_1, b_5, b_3)$$
$$G = (a_5, a_4, a_{12})(b_6, b_3, b_{11})$$
$$H = (a_6, a_8, a_{12})(b_6, b_8, b_{12})$$

Looking at these generators, we see b_n's are never sent to a_m's and vice versa. Thus, a sticker labeled b_n can never be sent to the position of a sticker labeled a_m, so the Dino Cube group is not transitive on the stickers. □

Designing a New Cube

One of the main goals of the project was to design an original twisty puzzle. The inspiration for a puzzle came from looking at the mastermorphix. The mastermorphix is a pyramid shaped twisty puzzle. Upon playing with a mastermorphix, an experienced Rubik's Cube solver will notice that it functions in the same way as the Rubik's Cube. In fact, the mastermorphix could be thought of as a misshapen Rubik's Cube.

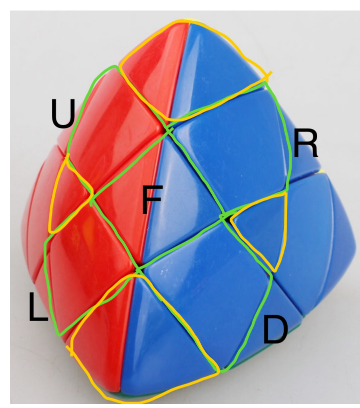


Figure 3: The mastermorphix is a shape-mod of a Rubik's Cube. the Edges around the F face are highlighted in green, and the corners in yellow.

Seeing such a puzzle brought about the question: If a cube shaped puzzle can be morphed into a pyramid shaped puzzle, can a pyramid shaped puzzle be morphed into a cube shaped puzzle? Of course, the mastermorphix could be changed back to the Rubik's Cube, but this is trivial – what about other pyramid shaped puzzles. The only other pyramid puzzle available to us was the Pyraminx, which functions very differently than the Rubik's Cube. After drawing several pictures trying to come up with a way to turn the pyraminx into a cube shaped puzzle, the sketches in figure 5 were achieved.

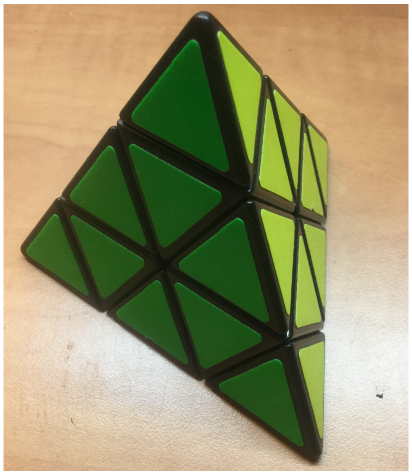


Figure 4: The Pyraminx is one of the more well known twisty puzzles

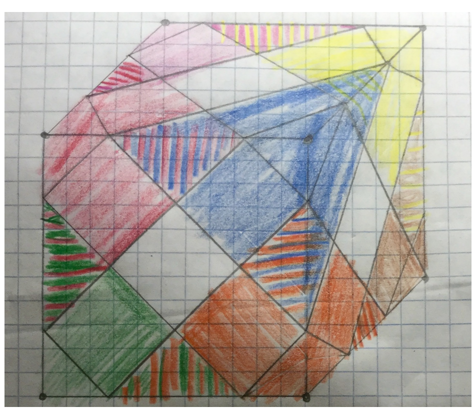
This design is similar to the pyraminx, though it has a key difference. The pyraminx turns around its four corners, but this puzzle turns around its 8 corners. The turns are similar – they each consist of a rotating a corner in place, and cycling three edges – but there are different numbers of them.

Despite the failure to turn the pyraminx into a cube shaped puzzle, this new puzzle was found to not yet exist, and so it was this design that was brought to life through the project.

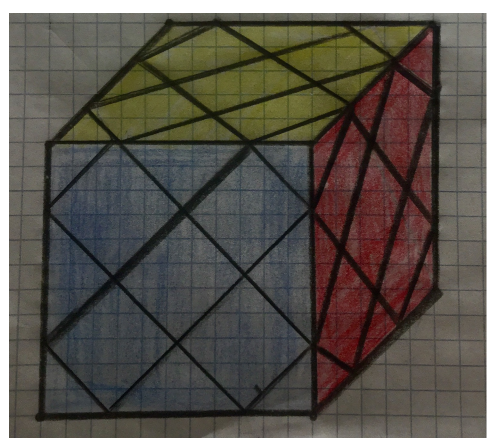
This puzzle, now called the Roffo Cube, consists of 8 corners, 12 edges, and 6 centers. A turn on the Roffo Cube means to rotate a corner clockwise 120 degrees, thus permuting the three edges adjacent to that corner. One important thing to note about the cube is that since the edges and corners cannot switch places, the edges and corners can be thought of separately.

Mathematically speaking, the *orbit* of a corner (all the places it can go) is itself, and the orbit of an edge is all the edges of the puzzle. It becomes clear that the edges of the Roffo Cube work in the same manner of the edges of the Dino Cube. In fact, the group of permutations on the edges via corner turns of the Roffo cube is *isomorphic* to (or the same as) the Dino Cube group.

Figure 5



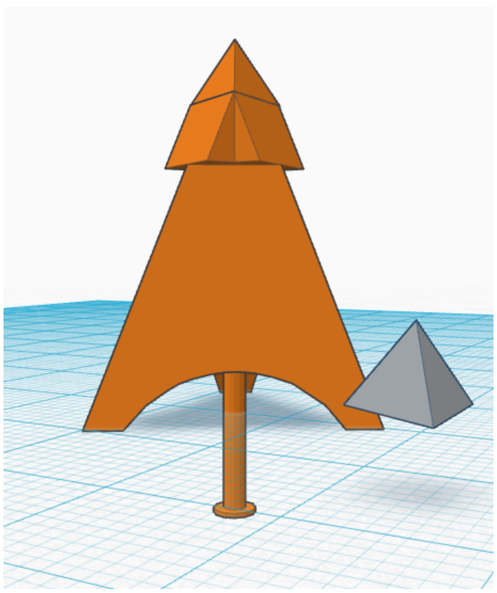
(a) Shown above are the cubies affected by each corner turn of the Roffo Cube. For example, turning the blue corner will move all of the three edges containing blue. Note that edges have two colors, since they are adjacent to two corners at any point in time.



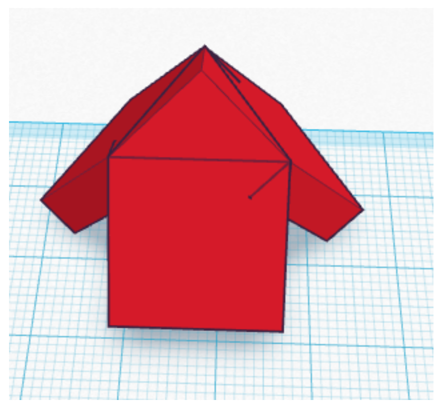
(b) Shown above is how the Roffo cube would look after being printed in black and properly stickered.

Corollary. (To Proposition 2)
The Roffo Cube group is not transitive on the stickers.

It is possible to explore the Roffo Cube group using mathematics without having a physical copy, however one of the goals of this project was to 3D print the puzzle. Tinkercad was used to design the puzzle. The design process involved some 3D geometry, especially for making the corner cubies, which had to have right angles between each pair of faces at the apex. After 10 or so weeks of designing, the cube was printed at Penfield library. This allowed for the design to be tested, changed and reprinted, and eventually the final version was printed via Shapeways.com.



(a) My first attempt at creating a corner piece.



(b) The outside of the final version of the corner pieces.

Figure 6: Beginning steps in Tinkercad.

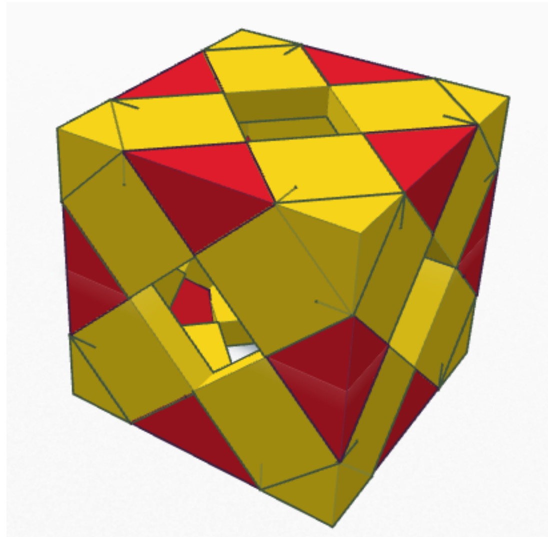


Figure 7: The final versions of the edges and corners before adding the internal mechanism.

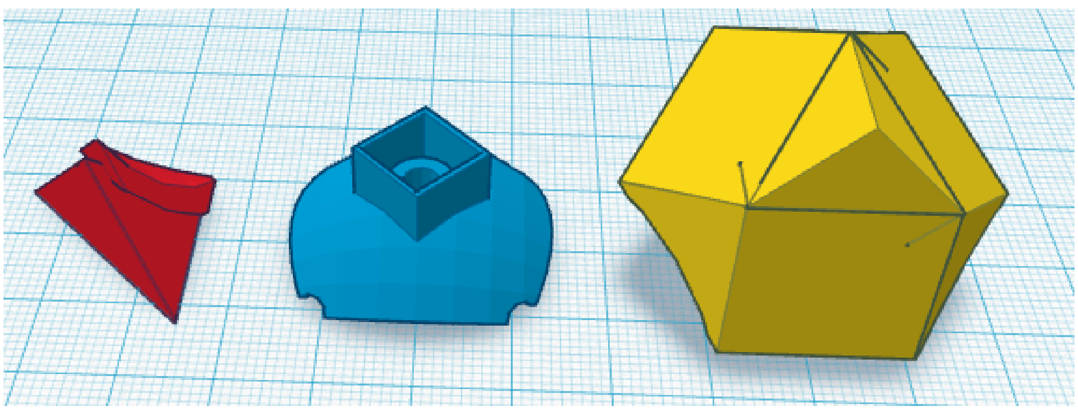


Figure 8: The final pieces, ready for 3D printing.

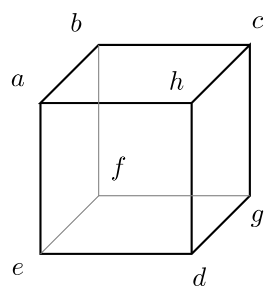
Corner Edge Independence

On the Roffo Cube, the edges and corners cannot permute with each other. Thus the Roffo Cube group may be a direct product of a group for the edges (the Dino Cube Group) and the group which describes the ways the corners can be permuted.

A given corner has three orientations in its fixed position, so the group of permutations on a single corner is $\mathbb{Z}_3$. Since there are 8 corners, each independent from the others, the group describing the 8 corners is really an 8-fold direct product of $\mathbb{Z}_3$, which we denote by $\mathbb{Z}_3^8$. One might begin to think that the Roffo Cube group is the direct product of the Dino Cube group, D, and the corner describing group, $\mathbb{Z}_3^8$, but there is a problem. To assume this is to assume that every permutation in the direct product $D \times \mathbb{Z}_3^8$ is a member of the Roffo Cube group, but this is not known to be the case. This would imply that one could conceivably solve the edges of the puzzle like a Dino Cube, then orient each corner separately.

In the end, we were able to answer this question using the physical cube, but before the cube was printed, we applied theory in an attempt to explore the problem. We looked for a way to turn just one corner of the cube without affecting the edges. Being able to do so would allow any permutation in the direct product to be executed by permuting the edges first (while ignoring the corners), and then permuting the corners without disturbing the edges. Thus finding such a permutation would be enough to prove that the Roffo Cube group is $D \times \mathbb{Z}_3^8$.

To start, we label the corners of the Roffo Cube via the following diagram. Here, the labels are the permutations achieved by rotating the corresponding corner clockwise.



To turn exactly one corner, without affecting any edges, we first endeavor to find a way to turn at least one corner, without affecting the edges. The following algorithm does just that:

$$hcbadgcb^2afba^2f^2gd^2f^2gfg^2f^2gfg^2.$$

The way this permutation effects the corners is equivalent to the way the corners are affected by $h(abc^2g^2)$. Since the edges are not affected, this shorter notation will be used to represent the larger one. Note that we could rotate the cube through any of its rigid motions, and we can find 24 different permutations which exhibit this behavior.

$h(abc^2g^2)$	$c(ab^2f^2h)$	$b(a^2ce^2h)$	$a(bcd^2h^2)$
$h(a^2b^2de)$	$c(a^2dh^2g)$	$b(c^2fgh^2)$	$a(b^2c^2ef)$
$h(cd^2e^2g)$	$c(bd^2fg^2)$	$b(aef^2g^2)$	$a(de^2f^2h)$

$d(a^2e^2fg)$	$e(b^2df^2g)$	$f(c^2deg^2)$	$g(d^2efh^2)$
$d(cf^2g^2h)$	$e(ad^2g^2h)$	$f(abd^2e^2)$	$g(bce^2f^2)$
$d(ac^2eh^2)$	$e(a^2bfh^2)$	$f(a^2b^2cg)$	$g(b^2c^2dh)$

If these permutations can be used to generate a single turn of a corner, then by symmetry it will be possible to turn every corner, implying the group which acts on the corners of this cube is $\mathbb{Z}_3^8$. If that is the case, then there are no less than 8 generators for the corner group of the Roffo cube (one permutation for each corner). Thus if it can be shown that these permutations can be generated by less than 8 permutations, they do not form $\mathbb{Z}_3^8$.

One can see

$$(h(abc^2g^2))(c(a^2dgh^2)) = bd.$$

It turns out that all 24 generators can be paired up in such a way, and when done, we have 4 permutations, each of which can be obtained by three different combinations of two generators. Also

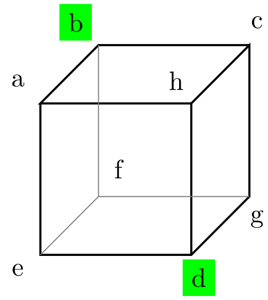
$$(h(abc^2g^2))(b(c^2fgh^2)) = ab^2cf.$$

Again, it turns out that we end up with 8 permutations of this form.

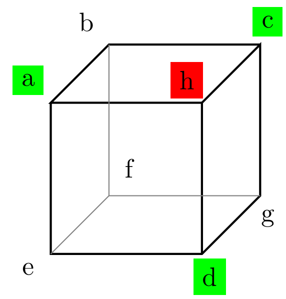
We have now found twelve more permutations, but for what purpose? The goal is to build a generating set which contains less than 8 elements, as this would show that the corner permutations do not form $\mathbb{Z}_3^8$. However, before looking at ways of reducing the twelve we have now, we must first convince ourselves that they indeed form a generating set for the group of permutations on the corners of the Roffo Cube.

It may help to give a more concrete example of what these are.

Figure 9

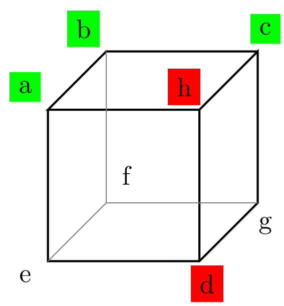


(a) The first simplified permutations just show us that we have the ability to rotate opposite corners on the cube together. For example, bd rotates corners b and d, which are opposite of each other (on the left). These will be referred to as *opposite corner permutations.*



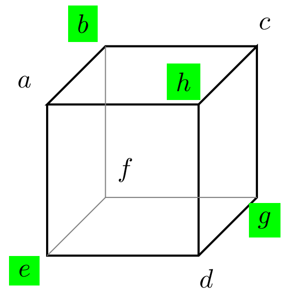
(b) The second simplified permutation consists of rotating a corner twice (red), and all 3 of its neighbors once (green). For example, in $acfb^2$, we see a, c and f are the neighbors of b and each of them turns once while b turns twice (on the right). These will be called *T permutations.*

If we combine bd with $acdh^2$, we get $a(bcd^2h^2)$. One can imagine taking the above two images and overlaying them on top of each other. Since green squares represent single turns, when two greens are combined they become a red, since they form a double turn. This is illustrated below.

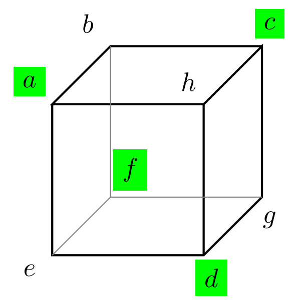


We also notice that we can combine $a(bcd^2h^2)$ with each of ce and ag to get $d(ac^2eh^2)$ and $c(a^2dgh^2)$. Likewise, we can take any of the eight T shape permutations and combine it with each of three opposite corner rotations to generate three more elements from the given generating set. That's 24 total permutations, and one can check that in fact they are the same 24 permutations in our generating set.

This shows that this set of 12 permutations is a smaller generating set for the group of permutations of the corners of the Roffo Cube.



$(begh)$



$(acdf)$

ag	a^2beh	ade^2f
bd	ab^2cf	bef^2g
ce	bc^2gh	$cdfg^2$
fh	d^2egh	$acdh^2$

By using two generators from the first generating set, we have

$$(h(abc^2g^2))(b(a^2ce^2h)) = b^2e^2g^2h^2.$$

Note that doing this twice gives $begh$. We also have

$$(f(c^2deg^2))^2(d(a^2e^2fg))^2 = acdf.$$

Now let's see what happens when we combine these with in our 12 element generating set.

$$(acdf)(ag)^2 = cdfg^2 \qquad (acdf)(ce)^2 = ade^2f \qquad (begh)(ag)^2 = a^2beh \qquad (begh)(ce)^2 = bc^2gh$$
$$(acdf)(bd)^2 = acb^2f \qquad (acdf)(fh)^2 = acdh^2 \qquad (begh)(bd)^2 = d^2egh \qquad (begh)(fh)^2 = bef^2g$$

Combining each of the two new permutations with all four of the opposite corner permutations gives all of the T permutations. Thus the set containing the two new permutations and the 4 opposite corner permutations is a generating set. But this set has 6 elements, which is less than 8, so the group generated by it cannot be $\mathbb{Z}_3^8$.

Thus using the algorithm I have found, it could not be shown that the Roffo Cube corner group is $\mathbb{Z}_3^8$, but this does not mean it is not. It is simply inconclusive. After this investigation the question remained unanswered, but having a functional Roffo Cube led us to the answer.

The Roffo Cube

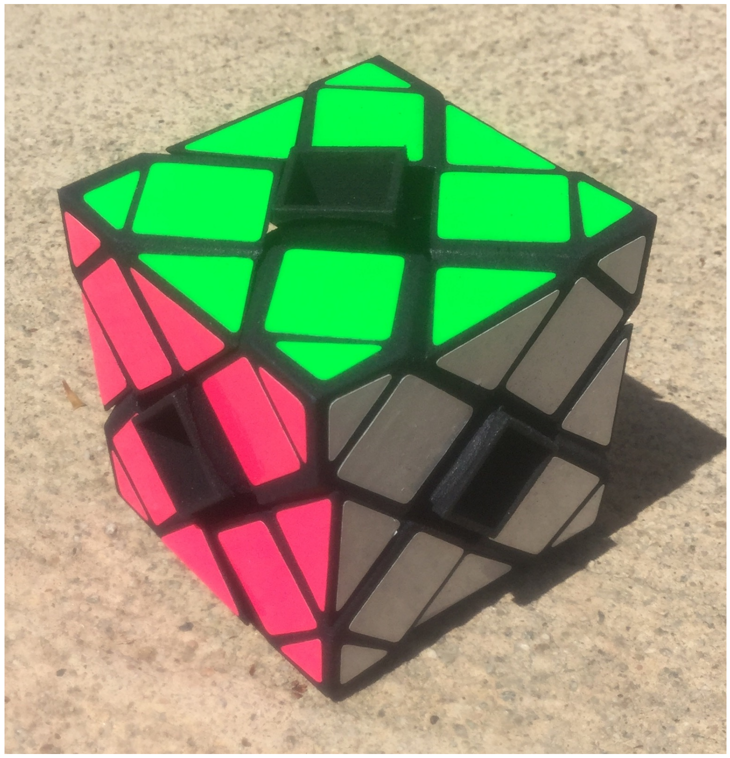
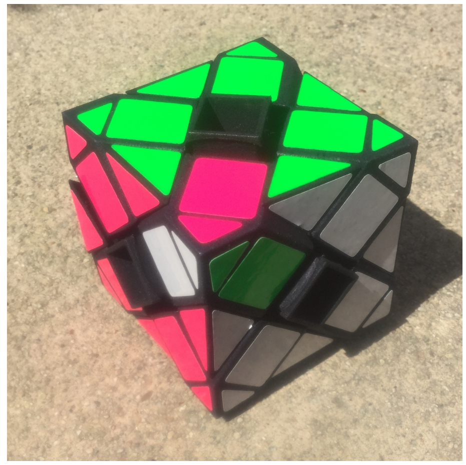


Figure 11: The final version of the Roffo Cube. The centers do not move, and they are not stickered, but they play a key role in holding the edges into the puzzle during a turn.

With the Roffo Cube complete, it became possible to explore the its properties by actually performing permutations. Upon experimenting with the puzzle for several minutes, I found the following algorithm (Notation from previous section):

$$(hah^{-1}a)^2$$

Execution of this permutation rotates the a corner while nothing else on the puzzle is effected, as seen on the following page.



This shows that the Roffo Cube Group is isomorphic to

$$D \times \mathbb{Z}_3^8.$$

Every permutation in the group has the form

$$k = (x, y_a, y_b, y_c, y_d, y_e, y_f, y_g, y_h)$$

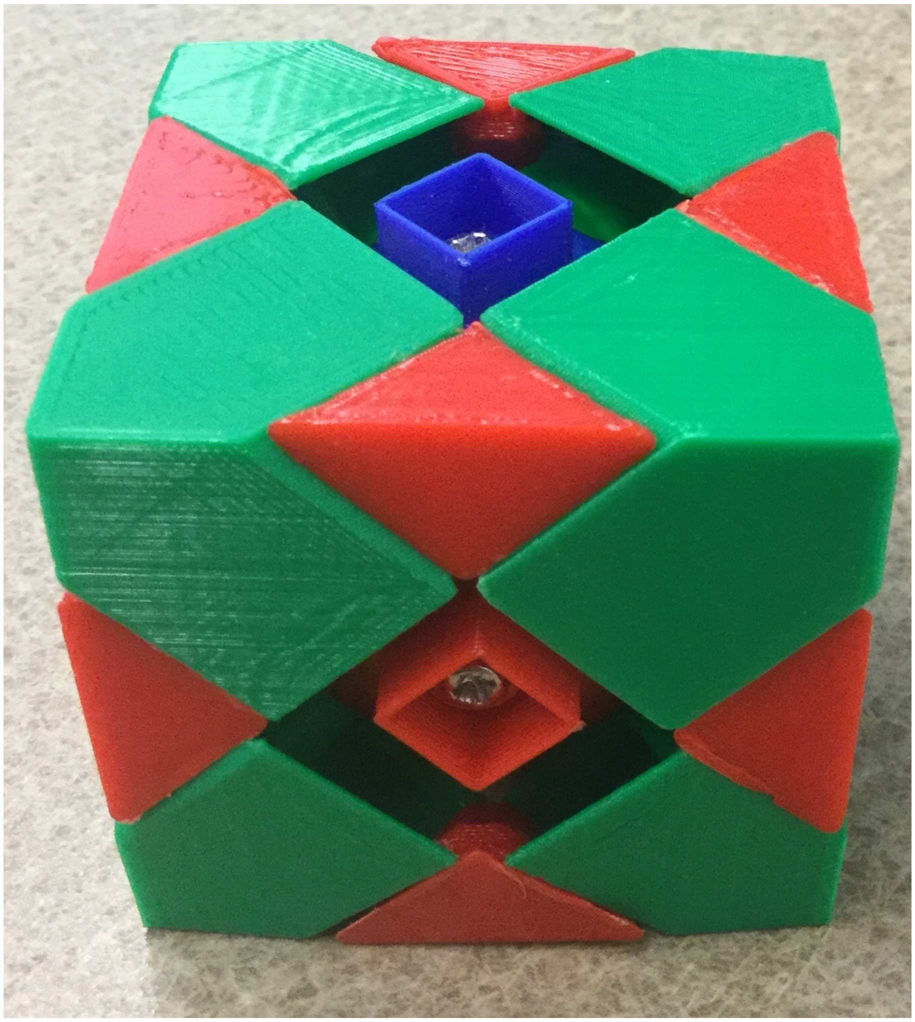
where x is the permutation on the edges and $y_a, y_b, ...y_h$ are the permutations on each of the corners a-h respectively. To perform a given scramble, execute x to permute the edges, then use the corner rotating permutation on each corner until the scramble is achieved.

Conclusions

Studying twisty puzzles with math can be difficult, but it is a very fun and effective way of exploring different concepts in group theory. From a cuber's perspective, learning group theory explains a lot of things which happen while playing with twisty puzzles, which before seemed to have no reason. From a mathematician's perspective, twisty puzzles appear to be very complex creatures and even things which act in simple ways can open up an entire world of complexity.

Designing a twisty puzzle is no easy task, but it is well worth it. It is a process which requires a lot of time and patience, but having an original twisty puzzle is a great accomplishment to be proud of.

The final copy, as well as prototypes, is kept in the SUNY Oswego Math Department.



A prototype of the Roffo Cube.